Silent interests and all-pay auctions

Kai A. Konrad*

August 25, 2005

Abstract

If firms compete in all-pay auctions with complete information, silent shareholdings introduce asymmetric externalities into the all-pay auction framework. If the strongest firm owns a large share in the second strongest firm, this may make the strongest firm abstain from bidding. As a consequence, equilibrium profits of both firms may increase, but the prize may be allocated less efficiently. The reverse ownership structure is also likely to increase the profits of the firms involved in the ownership relationship but without these negative efficiency effects.

Keywords: all-pay auctions, complete information, externalities, silent minority shareholdings, ownership structure.

JEL classification numbers: D44, L11, L41

---

*Social Science Research Center Berlin (WZB) and Department of Economics at Free University of Berlin. Correspondence address: Kai A. Konrad, WZB, Reichpietschufer 50, D-10785 Berlin, Germany. E-mail: kkonrad@wz-berlin.de. I thank Aron Kiss, Dan Kovenock, Johannes Münster and two anonymous referees for very helpful comments and the German Science Foundation (DFG) for support (grant SFB-TR 15). The usual caveat applies.

1 Introduction

It is common for a firm to own minority shares in another competing firm, and these shareholdings are often small and not sufficient to give the firm that owns them formal control rights.[1] As discussed, for instance, in Reynolds and Snapp (1986), shareholdings of firm A in firm B may soften competition and increase industry profits, but most of these benefits go to outsiders or to firm B. In Cournot markets, firms therefore have little or no incentive to acquire shares in a market competitor (Reitman 1994). Investing unilaterally in silent shareholdings can be profitable, depending on the shape of cost functions and the type of competition, as has been shown by Farrell and Shapiro (1990), Flath (1991) and Reitman (1994), but when the owners of a firm A acquire a partial interest without control rights in a firm B, there is a general tendency for this to benefit the owners of firm B and other firms more than the owners of firm A. Minority shareholdings, or even cross-ownership holdings, are also mixed blessings with respect to the sustainability of tacit collusion (Malueg 1992).[2]

Even though competition in auctions is an important part of modern business interaction, the role of minority shareholdings is less well understood in such auction markets. Ettinger (2002, 2003) and Dasgupta and Tsui (2004) are among the few contributions. However, they consider auctions, not all-pay auctions and do not address the issue of whether acquiring shares in another firm is individually rational and profitable in such a context.[3] Chillemi

[1] See, e.g., Gilo (2000) for some empirical evidence.

[2] Minority shareholdings and the incentives an individual firm has to acquire such holdings must be distinguished from the problem of forming a coalition in terms of mutual cross-holdings or joint ventures, where both sides share in the benefits of reduced competition in a symmetric fashion. Mutual cross-holdings are much more likely to be profitable for all participants for a broad range of market games (see, e.g., Kwoka 1992).

[3] R&D processes often take a special form of a contest. For an analysis of the role of complementarity of research inputs in non-cooperative joint ventures and a literature survey see Anbarci, Lemke and Roy (2002).

(2004) considers auctions and all-pay auctions with incomplete information with cross-holdings that fulfill a specific symmetry requirement. Structurally related problems emerge in auctions with externalities more generally.[4] All-pay auctions with externalities in bid payments are considered by Goeree et al. (2005) and Engers and McManus (2004). They compare auctions and all-pay auctions in which the prize that is auctioned is private, but the bids that are paid are used to finance a public good.

All-pay auctions are the mode of competition between firms when they compete for market shares by advertizing and other marketing activities (see, e.g., Schmalensee 1976), when firms compete for large projects and spend resources, for instance, in beauty contests, like in architecture, in the competition for spectrum rights in telecommunications in some countries (see, e.g., Börgers and Dustmann 2003, p. 223 for a country survey) or in markets with strong network externalities in which firms compete for a de facto natural monopoly. This type of competition has been discussed in more detail, e.g., in Huck, Konrad and Müller (2001) and Konrad (2000).

Here competition between firms will be analyzed if the mode of competition is an all-pay auction with complete information and if one firm has a silent minority interest in one of the other firms. It turns out that the acquisition of minority shareholdings in a competitor can be in the interest of both the owners of the buying firm and the majority owners of the firm in which minority shareholdings are acquired. The firm with the highest valuation of winning the auction prize may find it profitable to purchase a minority share in the firm with the second highest valuation of winning, and, given these shareholdings, may then abstain from bidding in the equilibrium. The results in this paper provide a partial solution to the more general problem of an all-pay auction with externalities in a framework with complete information.[5]

[4] A key contribution to auction theory with externalities is Jehiel, Moldovanu and Stacchetti (1996).

[5] For all-pay auctions with externalities only partial results exist. The symmetric case in which the prize is a public good to some group of competitors that has been considered

In section 2 the general structure of the problem that is analyzed here is described in more detail. The benchmark equilibrium without shareholdings as in Baye, Kovenock and deVries (1996) is discussed in section 3. Minority shareholdings and their profitability and welfare properties are analyzed in section 4. Section 5 concludes.

2 The problem structure

Consider a set N of firms $i = 1, ...n$ which compete for winning a prize in an *all-pay auction* with *completely informed bidders*. The prize can, for instance, be winning a business contract in a contest, and the firm's valuation v_i denotes the *operating business surplus* which this firm generates from receiving and fulfilling this business contract. Let firms be sorted and numbered such that $v_1 \geq v_2 > ... > v_n$.[6]

Each firm i can make a bid $x_i \geq 0$. It has to pay this bid in full, independent of whether the firm wins the prize or not. The bid in an all-pay auction is like an up-front effort that is irreversibly lost once it is chosen. In what follows, the 'bid' will sometimes be called 'the firm's effort' in order to emphasize its non-contingent nature.

The outcome of the all-pay auction is determined as follows. The prize is awarded to the firm that makes the highest effort. If several firms make the same highest effort, the prize is awarded to each of these firms with the same probability. More formally, let M denote the set of firms j for which $x_j \geq x_k$ for all $k \in N$, and let $\#M$ be the number of firms in M. The probability that i wins the contest is a function of all contestants' efforts, and is denoted

in Baik, Kim and Na (2001), is one example.

[6]Cases with further non-strict inequalities, particularly with $v_2 = v_3$, lead to a multiplicity of equilibria that is avoided here for simplicity. For a complete description of the equilibria for other non-strict inequalities in the all-pay auction with complete information and without externalities see Baye, Kovenock and deVries (1996).

as

$$p_i = p_i(x_1, x_2, ...x_n) = \frac{1}{\#M} \text{ if } i \in M \text{ and } p_i = 0 \text{ otherwise.} \tag{1}$$

Given the operating surplus v_i, bid cost x_i and the contest success function p_i, we can define the *operating profit* of a firm as

$$\pi_i = p_i v_i - x_i. \tag{2}$$

If firms do not own shares in another firm, this operating profit is what the firm maximizes and what is distributed among the shareholders.

Firms may own or acquire shares in other firms. For instance, firm i may own a minority share θ_{ij} in firm j. To be 'silent', the minority share should be limited to $\theta_{ij} \in [0, 1/2)$.[7] One could also consider cross ownership or the equilibrium ownership structure of shares more generally, but, in what follows, I restrict attention to $\theta_{kl} \equiv 0$ for all firms $k \neq i$ and $l \neq j$. I ask how this transaction changes the nature of the all-pay auction equilibrium. If firm i owns a share θ_{ij} in firm j, this changes the overall profit of this firm to

$$W_i = \pi_i + \theta_{ij}\pi_j. \tag{3}$$

Each firm will generally maximize this *overall firm profit* W_i, which, for $\theta_{ij} = 0$ reduces to $W_i = \pi_i$.[8]

A silent share ownership θ_{ij} will be called *mutually profitable* if a change in the ownership structure from $\theta_{ij} = 0$ to this $\theta_{ij} > 0$ increases the sum of

[7] What makes this question interesting is that a merger or take-over may be subject to merger control in situations in which obtaining a silent minority share is not.

[8] One could also go one step further and consider the private ownership structure of firms and how the private owners may induce managers not to maximize firm profit but some other objective function (see, e.g., Bolle and Güth 1992). In line with the standard assumptions in the literature on firm crossholdings, however, I will assume firms to maximize their own overall profit. This is in line with full rationality, for instance, if each firm is controlled by majority shareholders who do not have cross-holdings in competitors of this firm.

operating profits of firms i and j. Mutual profitability is a sufficient condition for a mutually beneficial trade of shares between the owners of firm i and the owners of firm j. Which silent share ownership is mutually profitable will be analyzed, but the market mechanism that is used to implement this trade is not considered.[9]

A second question will be whether the silent share ownership increases or decreases the expected value that is generated from the equilibrium allocation of the prize. The value

$$V = \frac{1}{v_1} \sum_{k=1}^{k=n} p_k v_k \tag{4}$$

will be called *measure of social value* of the equilibrium allocation of the prize. The maximum social value is reached if $V = 1$, i.e., if the firm with the highest valuation v_1 receives the prize with probability 1. Social value V will generally be a function of θ_{ij}, as ownership shares may change the equilibrium win probabilities p_i. This social value is also an appropriate measure of social welfare in cases in which effort is simply a transfer.

3 The equilibrium for $\theta_{ij} = 0$

If $\theta_{ij} = 0$, firms compete in an all-pay auction without externalities and each firm i maximizes π_i as in (2). The equilibrium has been fully described by Baye, Kovenock and deVries (1996) for any type of prize structure. The result applied to the firms context here is stated as

Proposition 1 (Baye, Kovenock and deVries 1996) *Let there be n firms with valuations $v_1 \geq v_2 > ... > v_n$ of the prize. Let each firm maximize (2) by a choice of non-negative effort x_i in an all-pay auction with complete*

[9] Depending on the dispersion of share ownership, such deals may also cause free rider problems, and the trading mechanism, together with the initial share ownership, will influence the share price for which the share trading may take place. The focus here is constrained to the question whether a mutually profitable deal exists.

information. The unique auction equilibrium is in mixed strategies as follows. Firm 1 chooses effort $x_1 \in (0, v_2]$ according to a cumulative distribution function

$$F_1(x_1) = x_1/v_2, \tag{5}$$

firm 2 chooses effort $x_2 \in [0, v_2]$ according to

$$F_2(x_2) = (1 - (v_2/v_1)) + x_2/v_1, \tag{6}$$

and firms $k = 3, ...n$ choose zero effort. The equilibrium expected efforts are

$$Ex_1^* = v_2/2 \ , \ \ Ex_2^* = \frac{v_2 v_2}{2v_1}, \ \textit{and} \ x_k^* \equiv 0 \ \textit{for} \ k = 3, ...n, \tag{7}$$

and the payoffs are

$$\pi_1^* = v_1 - v_2 \ \ \textit{and} \ \ \pi_k^* = 0 \ \textit{for all} \ \ k = 2, ...n. \tag{8}$$

This result has been used in many applications. It is well known by now, and a proof will not be repeated here. The equilibrium of the full-information all-pay auction is in mixed strategies. Only the two top firms (the ones who value winning the auction most highly) participate in the auction and make positive bids. They randomize their effort uniformly in the interval between zero and v_2, and choose a density that makes the only rival bidder who actively participates just indifferent with respect to all effort choices in this interval. The solution is characterized by a mass point at zero effort for the player with the second highest valuation v_2. The solution translates into expected effort choices and payoffs that are fairly simple and depend only on v_2 and on the difference between v_1 and v_2. In particular, only the player who values winning the most gets a positive payoff, and this payoff is equal to the difference $v_1 - v_2$ between this player's own valuation of winning and the second highest valuation of winning. The allocation of the prize in the all-pay auction with complete information is inefficient, at least in expectation, unless $v_1 = v_2$. The social value is $V = 1 - \frac{v_2}{2v_1} \frac{v_1 - v_2}{v_1}$. A key property of this equilibrium for explaining the results in what follows is that an increase in v_1 does not change firm 1's equilibrium strategies, but changes player 2's bidding behavior and increases the probability that player 2 bids zero.[10]

4 Minority ownership

Consider now a minority shareholding $\theta_{ij} > 0$ of firm i in firm j for one firm i and one firm j. The ownership share of firm i in firm j changes the valuation firm i attributes to the prize compared to the operating surplus, and the resulting valuation depends on the bidding behavior of other firms. Let $F_k(x)$ be the cumulative distribution function of bids by firm k. To illustrate, adopt for a moment the tie-breaking rule that i wins if $x_i = \max_{k \in N}\{x_k\}$, and let j have no mass point at x_i. Then (3) for a given x_i can be rewritten for some given bid level x_i as

$$W_i(x_i) = \prod_{k \notin \{i,j\}} F_k(x_i) F_j(x_i) v_i - x_i + \theta_{ij} v_j \int_{x_i}^{\infty} \prod_{k \notin \{i,j\}} F_k(x_j) dF_j(x_j) - \theta_{ij} E(x_j), \tag{9}$$

where the first two terms are i's operating profit and the second two terms are i's profit from silent share ownership. An increase in x_i has a direct effect via i's operating profit and an indirect effect on i's overall profit of $\frac{\partial W_i}{\partial \pi_j} \frac{\partial \pi_j}{\partial x_i} = -\theta_{ij} v_j \prod_{k \notin \{i,j\}} F_k(x_i) dF_j(x_i)$.

A first result is

Proposition 2 *If $\theta_{ij} > 0$ with $i < j$ and $j \geq 3$ or if $\theta_{ij} > 0$ in firm $j < i$ for $i \geq 3$, the bidding strategies as in Proposition 1 constitute an equilibrium. Such shareholdings do not change operating profits or overall profits of firms in the equilibrium.*

[10] Goeree, Anderson and Holt (1998) discuss the comparative static properties of all-pay auctions with complete information. These properties motivate them to consider 'noisy' players, as in Goeree, Anderson and Holt (1998) and in Anderson, Goeree and Holt (1998).

Proof. Note that $W_k = \pi_k = p_k v_k - x_k$ for all $k \neq i$, and $W_i = p_i v_i - x_i + \theta_{ij}[p_j v_j - x_j]$. Suppose all firms anticipate that all other firms choose the candidate equilibrium bid strategies as in Proposition 1. All firms $k \neq i$ maximize $\pi_k = p_k v_k - x_k$. Then the candidate equilibrium strategies in Proposition 1 are the optimal replies for all $k \neq i$. Moreover, (5) and (6), together with $F_k(0) = 1$ for all $k \in \{3, 4, ...n\} \backslash \{i\}$, implies that $p_k = 0$ for $k \geq 3$. Therefore, W_i also reduces to $W_i = p_i v_i - x_i = \pi_i$, making F_i as in Proposition 1 an optimal reply. ∎

Intuitively, if firms own shares in inactive firms, or if inactive firms own shares in other firms, this does not change the incentives of active firms. Interesting implications of a firm's ownership shares in another firm can be expected only for ownership shares θ_{12} and θ_{21}. The case $\theta_{12} > 0$ is considered first.

Proposition 3 *(i) if* $0 < \theta_{12} < (v_1 - v_2)/v_2$ *then an equilibrium exists with* $\pi_k = W_k = 0$ *for all* $k \geq 2$ *and*

$$W_1 = \ v_1 - v_2 - \theta_{12} \frac{(v_2)^2}{2(v_1 - \theta_{12} v_2)}.$$

Share ownership in this interval is not mutually profitable. (ii) if $\theta_{12} \in (\frac{v_1 - v_2}{v_2}, \frac{v_1 - v_3}{v_2})$*, then an equilibrium exists with* $\pi_k = W_k = 0$ *for all* $k \geq 3$*,*

$$W_2 = \pi_2 = \ (1 + \theta_{12}) v_2 - v_1$$

and

$$\pi_1 = 0 \ \textit{and} \ W_1 = \theta_{12}(v_2 - \frac{v_1 - \theta_{12} v_2}{2}).$$

Share ownership in this interval is mutually profitable iff $\theta_{12} v_2 > 2(v_1 - v_2)$*.*

A proof is in the Appendix. Intuitively, if θ_{12} is small as in (i) then the nature of the equilibrium is not changed, compared to $\theta_{12} = 0$. Only firms 1 and 2 compete against each other. Firm 1's valuation of winning equals the difference between its own operating surplus v_1 and $\theta_{12} v_2$ (what it gets from losing), compared to $\theta_{12} = 0$. Firm 2 has the same prize of winning, but the prize for firm 1 is reduced. The difference between valuations of winning still determines the prize for the firm with the higher valuation, the firm with the second highest valuation competes actively but makes zero profit, and all other firms prefer to be inactive. If θ_{12} is sufficiently large to make $v_3 < v_1 - \theta_{12}v_2 < v_2$ as in (ii), when firms 1 and 2 compete, firm 2 still values the prize of winning at v_2. However, playing against firm 2, firm 1 gains only $v_1 - \theta_{12}v_2$ if the prize is awarded to 1 and not to 2, and this net prize of winning is so small that firm 2 values winning the prize more highly than firm 1. The firms switch ranks as regards prize valuations. As known from Proposition 1, the rank and the differences in valuations determine payoffs. Firm 2 receives a positive operating profit and firm 1 receives an operating profit of zero, and a positive overall profit, due to its shareholdings in firm 2.

Proposition 4 *If $\theta_{12} > (v_1 - v_3)/(v_2 - (v_3/2))$ then an equilibrium exists in which only firms 2 and 3 are active, and with $\pi_k = W_k = 0$ for all $k = 2, 4, 5, ...n$, $\pi_1 = 0$, $W_2 = \pi_2 = v_2 - v_3$ and $W_1 = \theta_{12}(v_2 - v_3)$. Moreover, an ownership share of this size is mutually profitable if and only if $v_1 - v_2 < v_2 - v_3$.*

A proof is in the Appendix. The most interesting result in Proposition 4 is the possibility that firm 1, which has the highest operating surplus from winning the auction, becomes passive and still makes higher profits than in the equilibrium with $\theta_{12} = 0$. Intuitively, without minority shareholdings, firm 1 and firm 2 may compete tightly and dissipate most, or all, of the operating surplus that can be obtained from winning the prize in the auction. If, instead, firm 1 successfully commits to not making bids, the difference between firm 2's and firm 3's valuations of winning describes the total rent that is allocated among firms, and, if this difference is larger than the difference in the valuations v_1 and v_2 of the prize for firms 1 and 2, total industry profit goes up for this θ_{12}, compared to $\theta_{12} = 0$. Even firm 1, and its owners, can benefit from this, as all this rent accrues in firm 2 and firm 1 receives a share in these profits that equals its ownership share in firm 2. The only problem is to make firm 1 credibly commit to bidding zero.

As discussed in Proposition 4, this will work only if firm 1's ownership share in firm 2 is sufficiently large, if the difference in the valuations of the prize for firms 1 and 2 is small, and if the difference in prizes between v_2 and v_3 is larger than the difference between v_1 and v_2, but also not too large. To see that the parameter range with $\theta_{12} < 1/2$ for which $\theta_{12} > (v_1 - v_3)/(v_2 - (v_3/2))$ holds is non-empty, note that this condition reduces to $v_3 > \frac{2}{3}v_2$ for $(v_1 - v_2) \to 0$ and $\theta_{12} = 1/2$.

The result is based on a mechanism that is explored in a paper by Baye, Kovenock and deVries (1993). They consider the problem of an auction designer to be whether to exclude the bidder who -in terms of this paper- has the highest operating surplus from winning the prize. The designer is interested in high total effort which is increasing in the valuation of winning of the bidder who has the second highest valuation, and decreasing in the difference between the highest and the second highest valuation of winning. In the absence of firms' ownership shares in other firms, excluding the bidder with the highest operating profit is worthwhile from the designer's perspective if it improves on competitive balance, and hence, total expected effort. In Proposition 4, the contestant with the highest operating surplus may choose an ownership structure that commits himself to not being active in the contest, if this sufficiently increases competitive imbalance. Unlike the contest designer who likes much effort, and hence, likes to increase competitive balance, the competitors themselves are interested in generating a high competitive imbalance, as a larger difference between them reduces contest effort.

Consider now the intermediate range of θ_{12} that is not covered by Propositions 3 and 4.

Proposition 5 *If* $\theta_{12} \in (\frac{v_1 - v_3}{v_2}, \frac{v_1 - v_3}{v_2 - (v_3/2))})$ *no equilibrium exists in which only* *two firms are active. For any equilibrium, a sufficient condition for θ_{12} in this range to be mutually profitable is $v_1 - v_2 < v_2 - v_3$.*

A proof is in the Appendix. Intuitively, if firm 1 loses, it still cares about whether firm 2 or some other firm (e.g., firm 3) wins the prize. Competing against firm 2, firm 1 is not very aggressive as some of the gains of firm 2 make firm 1 also gain, due to the shareholdings. However, competing against firm 3, firm 1's stake is larger and equal to the full valuation v_1 of the prize. The fact that firm 1 is not indifferent to whom it competes with generates an additional discontinuity in the problem and makes it difficult to address the question of existence, or to characterize an equilibrium. An intuition for why the minority share is mutually profitable in an equilibrium if $v_1 - v_2 < v_2 - v_3$ is based on the insight that bids higher than $x = v_3$ will not occur in the equilibrium, which yields some lower limit for firm 2's operating profits.

So far only profitability aspects have been considered.

Proposition 6 *Consider $\theta_{12} > 0$: In the equilibria that are characterized in Proposition 3 and 4, the social value $V(\theta_{12}) < V(0)$.*

Proof. The result follows from $p_1(\theta_{12} = 0) > p(\theta_{12} > 0)$ for all $\theta_{12} > 0$ and $p_1(\theta_{12} = 0) + p_2(\theta_{12} = 0) = 1 \geq p_1(\theta_{12} > 0) + p_2(\theta_{12} > 0)$ for all $\theta_{12} > 0$. ■

If firm 1 owns a stake in firm 2, this makes firm 1 a less aggressive bidder, up to the point where firm 1 becomes even fully inactive. Generally, this makes it less likely that firm 1 which has the highest v_1 wins, and, once only firms 2 and 3 are active, it becomes possible that firm 3, that values the prize by even less than firm 2 does, will win the prize.

The case $\theta_{21} > 0$ is considered next.

Proposition 7 *For $\theta_{21} \in [0, \min\{\frac{v_2 - v_3}{v_1}, \frac{1}{2}\})$ an equilibrium exists with the following properties: only firms 1 and 2 make positive bids. $W_1 = \pi_1 = v_1 - (v_2 - \theta_{21} v_1)$, $W_2 = \theta_{12}\pi_1$, $\pi_2 = 0$, $W_k = \pi_k = 0$ for all $k > 2$. Such minority shareholdings are mutually profitable. Moreover, social value is continuous at $V(0)$ and increasing in θ_{21} on this interval.*

A proof is in the Appendix. If firm 2 owns some shares in firm 1, and if the ownership share is small enough, firms 1 and 2 will remain to be the only firms that are active in the equilibrium. Hence, all other firms' operating profits are zero. Moreover, as long as firm 1 and firm 2 are the only active firms, firm 2's ownership share in firm 1 reduces the overall value which firm 2 attributes to winning, as firm 2 receives $\theta_{21}v_1$ even if firm 2 loses, and v_2 if it wins. The overall value that firm 1 attributes to winning is unchanged. Accordingly, as regards the bidding incentives, the share ownership in firm 1 is very similar to a reduction in firm 2's valuation of winning. It increases the difference in the two firms' overall valuations of winning and increases firm 1's profits. However, as firm 2 now owns a share in firm 1's profits, firm 2 receives a share in firm 1's operating profit. Also, the increase in the difference in overall valuation of winning makes it more likely that firm 1 wins the prize, and this increases V.

For small ownership shares, this reasoning applies monotonically for any further increase in the ownership share. However, as the difference in valuations increases, the overall valuation of winning for firm 2 becomes smaller and smaller. Once it drops below v_3, the nature of the equilibrium changes. Firm 3 becomes an active player. At least, the equilibrium that underlies Proposition 7 and is characterized in the proof of the proposition no longer continues to exist. Given the candidate equilibrium strategies (19) and (20), firm 3 would not remain inactive, but would bid higher than $v_2 - \theta_{21}v_1$.

There is again an intermediate range in which the equilibrium cannot be one in which only two players are active, and it is difficult to obtain a closed form solution for an equilibrium in this range. However, the mutual profitability of shareholdings θ_{21} can again be limited from below:

Proposition 8 *The sum of operating profits of firms 1 and 2 in an equilibrium with $\theta_{21} > (v_2 - v_3)/v_1$ is at least equal to $v_1 - v_3$.*

Again, the proof is in the Appendix. The equilibrium can be fully characterized as soon as θ_{21} becomes sufficiently large and also yields an intuition for the profitability result in Proposition 8:

Proposition 9 *If*

$$(v_1 - \frac{v_3}{2})\theta_{21} > v_2 - v_3 \text{ and } v_2 > \theta_{21}\frac{v_1}{2}. \tag{10}$$

then an equilibrium exists in which only firms 1 and 3 are active. The sum of overall profits for firms 1 and 2 are $v_1 - v_3$, making such shareholdings mutually profitable. Social value $V(\theta_{21})$ is smaller than $V(0)$ iff $v_3(v_1 - v_3) < v_2(v_1 - v_2)$.

As shown in the proof in the Appendix, firm 2 may become inactive in the bidding process if it owns a sufficiently large share in firm 1. Such shareholdings reduce the overall value that firm 2 attributes to winning vis-a-vis firm 1, and may reduce this value sufficiently to make firm 3 an active bidder. As a consequence, only firms 1 and 3 compete actively, and firm 2 enjoys some passive income from share ownership. This is profitable from the joint perspective of firms 1 and 2 as it increases firm 1's operating profit to $v_1 - v_3$, and keeps firm 2's operating profit at zero. However, when firm 2 becomes inactive, firm 3 becomes active. This has a negative impact for firm 2: with all firms except firm 1 and firm 2 inactive, if firm 2 does not win, firm 1 wins and firm 2 receives some shareholdership returns. Once firm 3 is active, firm 3 may also win, in which case firm 2 does not receive shareholdership returns. The condition (10) makes sure that the shareholdership returns from ownership in firm 1 are sufficiently large in the equilibrium in which only firm 1 and firm 3 are active so as to make it not attractive for firm 2 to make a positive bid.

5 Conclusions

Minority shareholdings can, but need not, increase firms' and industry profit in markets of the all-pay auction type. Particularly if the silent shareholdings a firm purchases in another firm increase the difference in valuation of winning for the firms who have the highest operating surpluses from winning the competition, the acquisition of such shareholdings is likely to benefit the owners of both firms that are involved in this acquisition. For a profit increase to emerge from silent ownership, the firm that has the highest operating surplus from winning the auction (the 'strongest' firm) may purchase shares in the firm that has the second highest operating surplus from winning or vice versa. The first type of share purchase typically decreases the efficiency of the prize allocation in the auction, the second type of share purchase can increase efficiency. Generally, silent share ownership is not neutral with respect to firm profits and efficiency, and firm profits and efficiency need not be aligned. Profitable share ownership deals may have different welfare effects, depending whether a stronger firm that has a higher operating surplus from winning the competition purchases shares in a weaker firm, or whether a weaker firm, i.e., a firm that has a smaller operating surplus from winning the competition purchases shares in a stronger competitor.

6 Appendix

Proof of Proposition 3. (i) The condition $0 < \theta_{12} < (v_1 - v_2)/v_2$ can also be stated as $v_1 - \theta_{12} v_2 \in (v_2, v_1)$. Consider the cumulative distribution functions $F_k(0) = 1$ for all $k = 3, ...n$,

$$F_1(x) = \begin{cases} \frac{x}{v_2} & \text{for} \quad x \in (0, v_2] \\ 1 & \text{for} \quad x > v_2 \end{cases} \tag{11}$$

and

$$F_2 = \begin{cases} (1 - \frac{v_2}{v_1 - \theta_{12} v_2}) + \frac{x}{v_1 - \theta_{12} v_2} & \text{for} \quad x \in [0, v_2] \\ 1 & \text{for} \quad x > v_2 \end{cases} \tag{12}$$

as candidate equilibrium strategies. Note that $\pi_k(x_k) = F_1(x_k)F_2(x_k)v_k - x_k < 0$ for all $x_k > 0$. Hence, $F_k(0) = 1$ is an optimal reply for $k = 3, ...n$.

Next, using $W_2 = \pi_2$ and inserting (11) in (2), $\pi_2(x_2) = \frac{x_2}{v_2}v_2 - x_2 = 0$ for all $x_2 \in [0, v_2]$ and equal to $v_2 - x_2 < 0$ for $x_2 > v_2$. Accordingly, any mixed strategy F_2 on the support $[0, v_2]$ is optimal for firm 2, in particular F_2 as in (12). Finally, inserting the candidate equilibrium cumulative distribution functions for $k = 2, ...n$, $W_1(x_1)$ becomes equal to $v_1 - v_2 - \theta_{12}\frac{(v_2)^2}{2(v_1-\theta_{12}v_2)}$ for $x_1 \in (0, v_2]$, and equal to $v_1 - x_1 - \theta_{12}\frac{(v_2)^2}{2(v_1-\theta_{12}v_2)}$ for $x_1 > v_2$, and hence, strictly smaller for $x_1 > v_2$ than for $x_1 \in (0, v_2]$. This makes any bid $x_1 \in (0, v_2)$ optimal and makes the candidate equilibrium strategy $F_1(x)$ as in (11) an optimal reply.

As regards mutual profitability, the sum of profits is $\pi_1 + \pi_2 = (v_1 - \theta_{12}v_2) - v_2 < v_1 - v_2$.

(ii) The condition $\theta_{12} \in (\frac{v_1-v_2}{v_2}, \frac{v_1-v_3}{v_2})$ can be stated as $v_1 - \theta_{12}v_2 \in (v_3, v_2)$. Consider the following cumulative distribution functions as candidate equilibrium strategies: $F_k(0) = 1$ for all $k = 3, ...n$,

$$F_1(x) = \begin{cases} 1 - \frac{v_1-\theta_{12}v_2}{v_2} + \frac{x}{v_2} & \text{for} \quad x \in [0, v_1 - \theta_{12}v_2) \\ 1 & \text{for} \quad x > v_1 - \theta_{12}v_2 \end{cases} \tag{13}$$

and

$$F_2(x) = \begin{cases} \frac{x}{v_1-\theta_{12}v_2} & \text{for} \quad x \in [0, v_1 - \theta_{12}v_2) \\ 1 & \text{for} \quad x > v_1 - \theta_{12}v_2. \end{cases} \tag{14}$$

Consider whether these are optimal replies to each other. Note that $W_k = \pi_k(x_k) = F_1(x_k)F_2(x_k)v_k - x_k < 0$ for all $x_k > 0$. Hence, $x_k = 0$ is an optimal reply for $k = 3, ...n$. Next, $W_2(x_2) = \pi_2(x_2) = (1 - \frac{v_1-\theta_{12}v_2}{v_2} + \frac{x}{v_2})v_2 - x = v_2 - [v_1 - \theta_{12}v_2]$ for all $x_2 \in [0, v_1 - \theta_{12}v_2]$ and smaller for any $x_2 > v_1 - \theta_{12}v_2$. Accordingly, any randomization F_2 on $[0, v_1 - \theta_{12}v_2]$ is optimal for firm 2, and this makes F_2 as in (14) an optimal reply. Finally, inserting the candidate equilibrium cumulative distribution functions for $k = 2, ...n$, $W_1(x_1) = \theta_{12}(v_2 - \frac{v_1+\theta_{12}v_2}{2})$ for $x_1 \in [0, v_2]$, and smaller for $x_1 > v_2$. This makes any randomization of x_1 on $(0, v_2)$ optimal and makes F_1 as in (13) an optimal reply.

As regards the sum of operating profits, $W_1(\theta_{12}) + (1-\theta_{12})W_2(\theta_{12}) = \pi_1 + \pi_2 = (v_1 - \theta_{12}v_2) - v_2 > v_1 - v_2$ iff $2(v_1 - v_2) < \theta_{12}v_2$. $\square$

Proof of Proposition 4. Firms $k = 2, ...n$ maximize $W_k = \pi_k = p_k v_k - x_k$. Suppose firm 1 abstains from bidding. Then the results from Proposition 1 apply to the set of firms $\{2, 3, ...n\}$ with firms 2 and 3 replacing firms 1 and 2. The unique equilibrium is then characterized by the mixed strategies $x_2 \in (0, v_3]$ distributed according to

$$F_2(x) = \frac{x}{v_3} \tag{15}$$

and $x_3 \in [0, v_3]$ distributed according to

$$F_3(x) = (1 - \frac{v_3}{v_2}) + \frac{x}{v_2} \tag{16}$$

and $x_i \equiv 0$ for all $i = 4, ...n$. For firm 1 the choice $x_1 \equiv 0$ is indeed optimal given (15) and (16). Firm 1's payoff from bidding $x_1 \geq 0$ is

$$\pi_1(x_1) = F_2(x_1)F_3(x_1)v_1 - x_1 + \theta_{12}\left[v_2 \int_{x_1}^{\infty} F_3(x_2)dF_2(x_2) - E(x_2)\right], \tag{17}$$

where $E(x_2)$ denotes the expected effort of firm 2. The last term is firm 1's share in firm 2's expected profit as a function of x_1, given that firms $k > 3$ choose $x_k = 0$. Substituting (15) and (16) into (17) yields

$$\pi_1(x_1) = \frac{x_1 v_2 - x_1 v_3 + (x_1)^2}{v_3 v_2} v_1 - x_1 + \theta_{12}\left[v_2 - v_3 - \frac{v_2 - v_3}{v_3}x_1 - \frac{(x_1)^2}{2v_3}\right]. \tag{18}$$

The term (18) is quadratic in x_1 and the quadratic term enters positively if $\frac{v_1}{v_2 v_3} > \theta_{12}\frac{1}{2v_3}$, which is always fulfilled, as $\theta_{12} < 1/2$ and $v_1 \geq v_2$. Hence, this function is strictly convex in x_1 and has its global maximum on the interval $[0, v_3]$ either at $x_1 = 0$ or at $x_1 = v_3$. From inserting $x_1 = 0$ and $x_1 = v_3$ in (17) one gets $\pi_1(0) = \theta_{12}(v_2 - v_3)$, and $\pi_1(v_3) = v_1 - v_3 - \theta_{12}v_3/2$. Accordingly, $x_1 = 0$ is optimal given (15) and (16) if $\theta_{12}(v_2 - v_3) > v_1 - v_3 - \theta_{12}v_3/2$, and this is equivalent to $\theta_{12} > (v_1 - v_3)/(v_2 - (v_3/2))$. This shows that $F_1(0) = 1$ is indeed an optimal reply.

The sum of the operating profits of firms 1 and 2 are $\pi_1(0) + \pi_2(0) = v_1 - v_2$, and $\pi_1(\theta_{12}) + \pi_2(\theta_{12}) = v_2 - v_3$ in this range of θ_{12}. $\square$

Proof of Proposition 5. Returning to the proof of Proposition 4, for $F_1(0) = 1 = F_k(0)$ for $k = 4, ...n$, if firm 1 does not make positive bids, the unique equilibrium cumulative distribution functions F_2 and F_3 are given by (15) and (16). However, $x_1 = v_3$ becomes superior to $x_1 = 0$ given these strategies if $v_1 - \theta_{12}v_2 \in ((1 - \frac{\theta_{12}}{2})v_3, v_3)$. This shows that there is no equilibrium in which only firms 2 and 3 make positive bids.

Consider next $x_k \equiv 0$ for all $k = 3, ...n$. A bid $\hat{x}_1 \in (v_3 - \epsilon, v_3)$ turns out to be strictly dominated by $x_1 = \epsilon$ for sufficiently small ϵ, whatever firm 2's strategy is. A choice $x_1 = \epsilon$ makes either firm 1 or firm 2 win. Hence, the payoff for firm 1 is

$$W_1(\epsilon) \geq \theta_{12}v_2 - \epsilon - \theta_{12}E(x_2).$$

If firm 1 choses $\hat{x}_1 \in (v_3 - \epsilon, v_3)$ then

$$W_1(\hat{x}_1) \leq v_1 - (v_3 - \epsilon) - \theta_{12}E(x_2).$$

Accordingly, $W_1(\epsilon) > W_1(\hat{x}_1)$ if $\theta_{12}v_2 - v_1 + v_3 > 0$ for sufficiently small ϵ. Moreover, $x_1 < v_3 - \epsilon$ also implies that $x_2 < v_3 - \frac{\epsilon}{2}$. Accordingly, firm 3 could make a positive payoff, for instance, by choosing $x_3 = v_3 - \frac{\epsilon}{4}$, and, hence, $F_3(0) = 1$ is not optimal. This shows that there is no equilibrium in which only firms 1 and 2 are active.

Finally, an equilibrium in which only firms 1 and 3 are active is not feasible. $W_2 = 0$ in such an equilibrium. Moreover, neither firm 1 nor firm 3 would make bids higher than v_3. Hence, firm 2 could make positive profit by making a bit in the interval $x_2 \in (v_3, v_2)$. Hence, a contradiction.

Note that firm 3 (or even firms $k > 3$) would never make bids higher than the operating surplus it could obtain from winning: $x_k \leq v_3$ for $k \geq 3$. Accordingly, the prize goes to firm 1 or firm 2 with probability 1 if at least one of them makes a bid equal to $v_3 + \epsilon$ for small positive ϵ. Further, if $F_k(x) = 1$

for all $x > v_3$ and $k \geq 3$, then firm 1 will never bid more than $v_3 + (\epsilon/2)$. Accordingly, for firm 2, $W_2 = \pi_2 \geq v_2 - (v_3 + \epsilon)$. Further, firm 1 maximizes $W_1 = p_1 v_1 - x_1 + \theta_{12}\pi_2 = \pi_1 + \theta_{12}\pi_2$. This has a lower bound equal to $\theta_{12}\pi_2$, for instance for a choice $x_1 = 0$. This implies that $\pi_1 = W_1 - \theta_{12}\pi_2 \geq 0$. In turn, $\pi_1 + \pi_2 > \pi_2 \geq v_2 - v_3 - \epsilon$. $\square$

Proof of Proposition 7. The strategies

$$F_1(x_1) = \frac{x_1}{v_2 - \theta_{21}v_1} \text{ with } x_1 \in [0, v_2 - \theta_{21}v_1], \tag{19}$$

$$F_2(x_2) = (1 - \frac{v_2 - \theta_{21}v_1}{v_1}) + \frac{x_2}{v_1} \text{ with } x_2 \in [0, v_2 - \theta_{21}v_1] \tag{20}$$

and $x_k \equiv 0$ for all $k = 3, ...n$ constitute an equilibrium for the following reasons. Firm 1 values winning by v_1 and attributes a value of zero to the event of any other firm winning the prize. For (20) and $x_k \equiv 0$ for all $k = 3, ...n$ firm 1's payoff is equal to $v_1 - (v_2 - \theta_{21}v_1)$ for any $x_1 \in (0, v_2 - \theta_{21}v_1]$, and smaller than this payoff for all other choices of x_1. Hence, (19) is an optimal reply for firm 1. It is assumed in Proposition 7 that $\theta_{21} < \frac{v_2 - v_3}{v_1}$. Hence, firms $k = 3, ...n$ value winning the prize by less than $v_2 - \theta_{21}v_1$. Given (19), these firms strictly maximize their payoff by a choice of effort $x_k \equiv 0$. It remains to show that (20) is an optimal reply for firm 2. Given (19) for firm 1 and $x_k \equiv 0$ for $k = 3, ...$, firm 2's payoff is $W_2(x_2) = F_1(x_2)v_2 + (1 - F_1(x_2))\theta_{21}v_1 - x_2 - \theta_{21}E(x_1)$, and can be rewritten as

$$W_2(x_2) = F_1(x_2)(v_2 - \theta_{21}v_1) - x_2 + \theta_{21}(v_1 - E(x_1)). \tag{21}$$

The third term on the right-hand side in (21) is a constant with respect to x_2, and is strictly positive, and it does not matter for firm 2's optimization choice. Hence, firm 2 behaves like a firm with a valuation of the prize that equals $(v_2 - \theta_{21}v_1)$. Given (19), this makes all $x_2 \in [0, v_2 - \theta_{21}v_1]$ yield the same payoff for firm 2, and this payoff is larger than for any other $x_2 \notin [0, v_2 - \theta_{21}v_1]$. Any mixed strategy on the support $[0, v_2 - \theta_{21}v_1]$ is then an optimal reply.

The equilibrium that is described by (19) and (20) and $x_k = 0$ for all other firms $k = 3, ...n$ has the properties outlined in the proposition. Aggregate payoff of firms 1 and 2 is $v_1 - (v_2 - \theta_{21}v_1)$, and this is strictly increasing in θ_{21}. This shows that shareholdings in this interval are mutually profitable. Moreover, V increases in θ_{21} as $\frac{\partial p_1}{\partial \theta_{21}} = -\frac{\partial p_2}{\partial \theta_{21}} > 0$. $\square$

Proof of Proposition 8. Consider $v_2 - \theta_{21}v_1 \in (v_3 - \theta_{21}\frac{v_3}{2}, v_3)$. Firms $k \geq 3$ will not bid higher than $x_k = v_3$. For this reason, any bid $\hat{x}_2 > v_3$ by firm 2 is dominated by a bid in the range $x_2 \in (v_3, \hat{x}_2)$. Accordingly, firm 2 will not make a bid higher than $v_3 + \epsilon$. In turn, this implies that firm 1 will not bid higher than $v_3 + 2\epsilon$. Accordingly, firm 1's operating profit is bounded from below by $v_1 - v_3 - \epsilon$. Moreover, as $W_2 = \pi_2 + \theta_{21}\pi_1$, this overall profit is bounded from below by $\theta_{21}\pi_1$, as this overall profit can be attained by $x_2 = 0$. Hence, π_2 is bounded from below by 0. Accordingly, the sum of operating profits must be bounded from below by $v_1 - v_3 - \epsilon$. Now consider $\epsilon \to 0$. $\square$

Proof of Proposition 9. Suppose that firm 2 does not make positive bids: $x_2 \equiv 0$. Then the equilibrium among the remaining firms is characterized in analogy to Proposition 1 as in Baye, Kovenock and deVries (1996), with firm 3 assuming the role of firm 2. The unique mixed strategies are characterized by

$$F_1(x) = \frac{x}{v_3} \text{ for } x \in [0, v_3] \tag{22}$$

and

$$F_3(x) = 1 - \frac{v_3}{v_1} + \frac{x}{v_1} \text{ for } x \in [0, v_3]. \tag{23}$$

For (22) and (23) $x_2 \equiv 0$ is indeed optimal for firm 2 if conditions (10) hold. Firm 2's overall payoff from bidding $x_2 \geq 0$ is

$$W_2(x_2) = F_1(x_2)F_3(x_2)v_2 - x_2 + \theta_{21}\left[v_1 \int_{x_2}^{\infty} F_3(x_1)dF_1(x_1) - \frac{v_3}{2}\right]. \tag{24}$$

Substitution of (22) and (23) into (24) yields

$$W_2(x_2) = \frac{x_2}{v_3}\frac{v_1 - v_3 + x_2}{v_1}v_2 - x_2 + \theta_{21}\left[v_1 - v_3 - (\frac{v_1 - v_3}{v_3}x_2 + \frac{1}{2}\frac{x_2 x_2}{v_3})\right]. \tag{25}$$

This expression is a quadratic function in x_2. The quadratic term enters with a positive sign, as $\frac{v_2}{v_3 v_1} > \theta_{21}\frac{1}{2v_3}$ by the second condition in (10). The function is, therefore, strictly convex and has its global maximum on the interval $[0, v_3]$ either at 0 or at v_3. From inserting $x_2 = 0$ and $x_2 = v_3$ in (24) one gets $\pi_1(0) = \theta_{21}(v_1 - v_3)$, and $\pi_1(v_3) = v_2 - v_3 - \theta_{21}\frac{v_3}{2}$. The choice $x_2 = 0$ is optimal if $\theta_{21}(v_1 - v_3) - v_2 + v_3 + \theta_{21}\frac{v_3}{2} > 0$, which can also be written equivalently as the first condition in (10). The firms' profits (including earnings from ownership shares) in this equilibrium become equal to $\pi_1 + \pi_2 = v_1 - v_3$ and $\pi_k = 0$ for all $k \geq 3$. Share ownership in this interval is mutually profitable for firms 1 and 2, as $v_1 - v_3 > v_1 - v_2$.

Social value is higher in the range of θ_{21} for which firm 2 becomes inactive than for $\theta_{21} = 0$ if $V(\theta_{21}) = 1 - \frac{v_3}{2v_1}\frac{v_1 - v_3}{v_1} > 1 - \frac{v_2}{2v_1}\frac{v_1 - v_2}{v_1} = V(0)$. This is true if $v_3(v_1 - v_3) < v_2(v_1 - v_2)$ $\square$

7 References

Anbarci, N., Lemke, R. and Roy, S., 2002. Inter-firm complementarities in R&D: a reexamination of the relative performance of joint ventures. International Journal of Industrial Organization 20, 191-213.

Anderson, S.P., Goeree, J.K. and Holt, C.A., 1998. Rent seeking with bounded rationality: an analysis of the all-pay auction. Journal of Political Economy 106(4), 828-853.

Baik, K.H., Kim, I.-G. and Na, S., 2001. Bidding for a group-specific public-good prize. Journal of Public Economics 82, 415-429.

Baye, M., Kovenock, D. and deVries, C., 1993. Rigging the lobbying process: an application of the all-pay auction. American Economic Review 83(1), 289-294.

Baye, M., Kovenock, D. and deVries, C., 1996. The all-pay auction with complete information. Economic Theory 8, 291-305.

Bolle, F. and Güth, W., 1992. Competition among mutually dependent sellers. Journal of Institutional and Theoretical Economics 148(2), 209-239.

Börgers, T. and Dustmann, C., 2003. Awarding telecom licences: the recent European experience. Economic Policy 36, 215-268.

Chillemi, O., 2004. Cross-owned firms competing in auctions. mimeo.

Dasgupta, S. and Tsui, K., 2004. Auctions with cross-shareholdings. Economic Theory 24(1), 163-194.

Engers, M. and McManus, B., 2004. Charity auctions. mimeo.

Ettinger, D., 2002. Bidding among friends and enemies. mimeo.

Ettinger, D., 2003. Efficiency in auctions with crossholdings. Economics Letters 80, 1-7.

Farrell, J. and Shapiro, C., 1990. Asset ownership and market structure in oligopoly. Rand Journal of Economics 21, 275-292.

Flath, D., 1991. When is it rational for firms to acquire silent interests in rivals?. International Journal of Industrial Organization 9, 573-583.

Gilo, D., 2000. The anticompetitive effect of passive investment. Michigan Law Review 99(1), 1-47.

Goeree, J.K., Anderson, S.P. and Holt, C.A., 1998. The war of attrition with noisy players. In: M. Baye (Ed.), Advances in applied microeconomics: Contests, 7, JAI Press, Greenwich, 15-29.

Goeree, J.K., Maasland, E., Onderstal, S. and Turner, J.L., 2005. How (not) to raise money. Journal of Political Economy (forthcoming).

Huck, S., Konrad, K.A. and Müller, W., 2001. Divisionalization in contests. Economics Letters 70(1), 89-93.

Jehiel, P., Moldovanu, B. and Stacchetti, E., 1996. How (not) to sell nuclear weapons. American Economic Review 86(4), 814-829.

Konrad, K.A., 2000. Trade contests. Journal of International Economics 51(2), 317-334.

Kwoka, J.E. Jr., 1992. The output and profit effects of horizontal joint ventures. Journal of Industrial Economics 40(3), 325-338.

Malueg, D.A., 1992. Collusive behavior and partial ownership of rivals.

International Journal of Industrial Organization 10, 27-34.

Reitman, D., 1994. Partial ownership arrangements and the potential for Collusion. Journal of Industrial Economics 42(3), 313-322.

Reynolds, R.J. and Snapp, B.R., 1986. The competitive effects of partial equity interests and joint ventures. International Journal of Industrial Organization 4(2), 141-153.

Schmalensee, R., 1976. A model of promotional competition in oligopoly. Review of Economic Studies 43(3), 493-507.