The Lifeboat Problem*

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Abstract

We study an all-pay contest with multiple identical prizes ("lifeboat seats"). Prizes are partitioned into subsets of prizes ("lifeboats"). Players play a two-stage game. First, each player chooses an element of the partition ("a lifeboat"). Then each player competes for a prize in the subset chosen ("a seat"). We characterize and compare the subgame perfect equilibria in which all players employ pure strategies or all players play identical mixed strategies in the first stage. We find that the partitioning of prizes allows for coordination failure among players when they play nondegenerate mixed strategies and this can shelter rents and reduce rent dissipation compared to some of the less efficient pure strategy equilibria.

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1 Introduction

When a ship sinks, passengers must hurry to find a lifeboat and, if the boat is crowded, there might be some competition over seats.[1] Still, even though there were more passengers than lifeboat seats on the Titanic, there were both lifeboats launched at less than full capacity and boats for which there was intense competition for seats. Indeed, ships sometimes sink much faster than the Titanic,[2] making the problem of the allocation of lifeboat seats an even more difficult, and sometimes less peaceful task.

The battle over seats in different lifeboats can be seen as a metaphor for a whole class of situations in which the players in a group partition themselves into smaller subgroups, and the members of each subgroup compete for one of several prizes that are awarded to some of the members of the respective subgroup. Competition may be intense in a subgroup if there are more contestants than prizes in this group, and competition is absent if prizes are abundant. Using the lifeboat metaphor, we call this game of endogenous self-partitioning followed by competition the *lifeboat problem*. We show in a symmetric version of this game that multiple subgame perfect equilibria exist and an equilibrium partitioning of players may be realized through both symmetric mixed strategies and asymmetric pure strategies. When there are more players than prizes in the aggregate, these endogenous choices can shelter rents that would be dissipated in a single large contest. The equilibrium payoff in a subgame perfect equilibrium with symmetric mixed strategies played in the first stage (as regards each player's choice of the set of prizes) can yield higher or lower rents than those arising in some of the equilibria employing pure strategies.

[1] The shortage of lifeboat seats has been frequently discussed in the most famous ship catastrophe, the sinking of the Titanic. In their empirical analysis Frey, Savage and Torgler (2009) analyze the determinants of who among the 2207 passengers of the Titanic survived. The sinking of the Titanic took several hours, and the process of evacuating the ship is reported as having been mostly civilized and fairly coordinated. See, for instance, the Report of the British Wreck Commissioner's Inquiry (1912) which can be both found at http://www.titanicinquiry.org/.

[2] Frey et al.(2009) note the case of the HMS Birkenhead which sank in 25 minutes in 1852. The American tranport ship The Antilles was struck by a torpedo in 1917 and sank in five minutes. 167 of the 237 persons on board were rescued. The cruiseferry MS Estonia sank in 1994 in a fifty-five minute period. Of the 938 passengers, it is estimated that up to 310 reached the outer deck and 160 managed to climb into liferafts or lifeboats. 138 passengers were rescued.

The lifeboat problem appears in a number of situations in which humans or other organisms must self-partition in a decentralized manner to engage in dissipative contests for prizes or other games in which competition dissipates rents among the competitors.

One important example is educational or professional choice in a world with specialization and complementary skills where scarcity premia accrue to players who choose a specialization which then turns out to be in excess demand. The empirical literature on education shows that both the enrollment rate in higher education[3] and the choices of particular fields exhibit considerable fluctuations over time,[4] rather than following a smooth pattern. The literature that addresses this problem has focused on the dynamics resulting from individuals who form their expectations on the basis of current labor market conditions or try to form rational expectations. One of the first approaches is Freeman (1975a,b). He suggested that students may follow a strategy that bases their decisions on observed wages, and concluded that this may cause patterns that look like cobb-web cycles. Siow (1984) and Orazem and Mattila (1991) also consider a dynamic approach, employing a rational expectations assumption. Borghans, de Grip and Heijke (1996) document the considerable mismatch between education choices and the labor market and attribute this to students' expecations that rest on current labor market conditions. This literature documents a considerable amount of mismatching and variation over time and suggests that mismatching is a loss. It does not focus on the problem of strategic interaction and the coordination problem these simultaneous and independent choices involve. The "lifeboat problem" that is analysed in this paper does not consider the dynamics of the matching problem of education and labor markets, but sheds light on the interdependence of the individuals' simultaneous decision problems and its potential for coordination failure (i.e., mixed strategy equilibria). It suggests that this coordination failure of student cohorts need not only have a detrimental effect, but may also shield some of their rents which would otherwise dissipate in the competition.

Examples exhibiting similar patterns appear in many other contexts. Professional tennis players need to choose among a set of simultaneous or overlapping tournaments, not knowing the choices of competitors with whom they would like to

[3]For a short survey of some of this evidence see Neugart and Tuinstra (2003).

[4]See, for instance, Eckstein, Weiss and Fleising (1988, p. 397) for electrical engeneering.

avoid competing. Contests among animals provide further examples. Single animals need to make a choice of nesting areas, where animals choose individually among distinct territories and overpopulation of a territory leads to a war of attrition.[5] Finally, the problem is related to enty games among firms if firms can choose between different business areas or product types, and if a choice of the same product type or business area generates fierce competition.[6]

For simplicity, we assume that players are identical from an *ex ante* standpoint, wish to obtain at most one prize, and that the contests are perfectly dissipative under homogeneity, as arises in a multiple prize all-pay auction or a multiple-unit pay-as-bid winner-pay auction with complete information. In these types of contests, if the number of identical prizes in a contest is greater than or equal to the number of competitors who compete for the prizes, everyone receives a prize with zero expenditure, but if the number of competitors is strictly greater than the number of prizes, dissipation is complete and the competitors receive no benefit in expectation. This notion of competition captures the idea that within a given environment in which the agents are competing, if competition is sufficiently "cut-throat" agents on the long side of the market gain nothing in expectation while those on the short side of the market collectively obtain all potential gains from trade.[7] In this context, the "fail-

[5] A critical aspect of these examples is that self-partition takes place across a set of contests and prizes that are fixed. That is, the matching problem faced is one-sided and not two-sided. Sometimes this is natural due to time-to-build on one side of the market. Lifeboats must be loaded on the ship before it sails. Specialized education programs or professions typically take many years to evolve. Wildlife habitats may take considerably longer to be developed. In other environments it may be more unclear whether one side of the market has a quasi-fixed nature. For instance, the choice of market platforms by buyers and sellers may involve two-sided self-partition. However if, for example, the sellers' side of the market involves large players who themselves establish platforms with which to interact with many small buyers, this may be viewed as a one-sided buyer problem with precommitted sellers.

[6] Models of entry games with a simple decision whether to enter a single market or stay out include Levin and Smith (1994), Elberfeld and Wolfstetter (1999), Vettas (2000), Cabral (2004) and Lu (2010). As pointed out by one reviewer, the "lifeboat" problem departs from much of this literature with endogenous entry, because in a "lifeboat" problem there are multiple parallel contests and no entry costs.

[7] This property arises not only in the Nash equilibria of many types of non-cooperative bidding games with identical players and complete information, but also in the core of certain cooperative coalitional games.

ure" to not fill all groups evenly turns out to be a huge benefit if there is an excess supply of players in the aggregate. In a grand contest with all players competing for all prizes, the payoffs are zero if there are more players than prizes. If the set of all prizes is partitioned into several sets of prizes, players who select into groups where there are at least as many prizes as players reap rents. Such rent generating partitioning arises both in pure strategy equilibria at the stage at which each player chooses a set of prizes, in which players coordinate in a self-enforcing fashion to not "spoil the pot" for a subset of rent earners, and in mixed strategy equilibria, in which "coordination failure" due to randomization in their choices yields stochastic rents that are symmetric in expectation across players.

Our analysis is related to a number of contest studies in the literature. A prominent paper in this line of research is the study of contest architecture by Moldovanu and Sela (2006). They study all-pay auctions with incomplete information and find that a grand contest for one big prize elicits more aggregate effort in expectation than either the same contest with the prize divided into many smaller prizes or a partition of the contest into multiple parallel contests in which both the set of players and the prize are equally divided. Moldovanu and Sela (2006) do not examine the self-partitioning of players into contests. Fu and Lu (2009) compare a situation in which all players compete with all other players ("the grand contest") with games in which the players are partitioned into subgroups. However, the formation of the subgroups is not part of the game. Piccione and Rubinstein (2007) consider a general equilibrium model of "equilibrium in the jungle" in which allocation is determined by a strength relation between pairs of players that is a complete ordering of the players. A special case of this model may be interpreted as the peaceful allocation of m prizes among n players (with $m > n$), but with heterogeneity of prizes and sequential allocation based on strength. Perhaps closest to our paper is Amegashie and Wu (2004), which examines a problem in which n players of strictly ranked abilities each choose between two contests with h and l prizes respectively, where there are at least two more players than prizes. Each of the h prizes in one contest are ranked identically and more highly by the players than each of the l identically ranked prizes in the second contest. Like in our framework, Amegashie examines a two stage game in which players simultaneously choose between contests and then play an all-pay auction with complete information within each contest. Amegashie provides a partial characterization of the set of subgame perfect equilibria with pure local strategies at the first stage of the game. He does not examine equilibria with mixed local strategies at the first stage. Moreover, he does not examine the question of how endogenous choice between different sets of prizes through pure or mixed strategies shelters rents, and how this outcome depends on the structure of the partition of the total number of prizes.

2 Group choice and the intra-group all-pay contest

We consider the following two-stage game. The set of (ex-ante symmetric) players is $N \equiv \{1, 2, ..., n\}$. In stage 1 each player chooses one element j of the set $B \equiv \{1, ..., b\}$. As one of the interpretations of the game is that the players are passengers on the Titanic who choose to go to one of the lifeboats, we can describe this choice as each player i chooses one lifeboat. Players' choices lead to a quasi-partition of players into b subsets $N_1, ..., N_b$, with $n_j \geq 0$ the number of players who have chosen the same life boat j, for all $j \in B$.[8] This completes stage 1. At stage 2 the players in each of the subsets N_j compete with each other in an all-pay auction with complete information and without noise. The rules of the game in this all-pay auction are as follows. Each $j \in B$ consists of k identical prizes which are valued equally by all players $i \in N_j$, and this value is normalized to 1. Each player $i \in N_j$ can win at most one prize. If the player wins no prize, this amounts to a prize value of zero. We may think of k seats in a life boat, where all players who receive a seat survive with certainty, whereas the players without a seat fare worse. The allocation of the k prizes among the n_j players follows the rules of an all-pay contest as in Barut and Kovenock (1998) among n_j symmetric players for $k < n_j$ identical prizes of size 1.[9] Each contestant chooses an

[8] A quasi-partition of the set N is a collection of sets $\{N_1, ...N_b\}$ such that $N_i \cap N_j = \varnothing$ if $i \neq j$ and $\cup_{i=1}^{b} N_i = N$ (see, for instance, Dunn and Hardegree 2001, p. 189). Note that a quasi-partition differs from a partition in that the sets N_i need not be nonempty, that is, some lifeboats may go unused. In the continuation we will sometimes refer to $(N_1, ...N_b)$ as a partition, even though the collection formally constitutes a quasi-partition.

[9] Clark and Riis(1998) examine the case of n players competing to win one of k prizes each of which players view as identical but for which players' values strictly differ. Clark and Riis note that when players value prizes identically multiple equilibria may arise. Barut and Kovenock (1998) examine the case where for any given prize, the players' values are identical, but this value may

effort e_i, and has a cost of effort $C(e_i)$, where $C(\cdot)$ is continuous and strictly increasing on $[0, \infty)$ with $C(0) = 0$ and $C(e) > 1$ for all $e > \bar{e}$, for some $0 < \bar{e} < \infty$.[10]

The subgame equilibrium payoffs of players in the group N_j can be extracted from Barut and Kovenock (1998), using their Theorem 2, setting the highest k prizes equal to one and all lower prizes equal to zero in their Theorem 2. Their result is stated as

Lemma 1 *(Barut and Kovenock 1998): The payoff of a player i in group N_j is equal to zero if $n_j > k$ and equal to 1 if $n_j \leq k$.*

The payoff for $n_j \leq k$ is immediately intuitive. For instance, if players $j = 1, ..., n_j$ approach seats $1, ..., n_j$, respectively, they face no competition, and this outcome is perfectly peaceful. If $n_j > k$, full dissipation occurs. This is less obvious but shown in Barut and Kovenock (1998).[11] Roughly, the result follows from the fact that the existence of more players than prizes implies that some player must earn an expected payoff of zero (at the lower bound of the union of the supports of the equilibrium strategies and, hence, in equilibrium) and such a player is willing to compete away any gains that other players might obtain by bidding strictly below one. This result is an important starting point for the analysis of the two stage game, as it gives us the unique equilibrium payoffs for all possible subgames in stage 2. We first consider subgame perfect equilibria in which players are able to coordinate in stage 1 when making their choices about which $j \in B$ they choose.

vary across the k prizes depending on the rank-order of the prize. In this environment, Barut and Kovenock characterize the complete set of Nash equilibria and show that equilibrium payoffs are unique although the set of equilibria could be quite large. The subgame employed in this paper is a special case of the Barut-Kovenock game where all k prizes are identical and valued identically by all players.

[10] Note that Barut and Kovenock (1998) assume linear costs of effort. However, reinterpreting $C(e_i)$ as the "bid" generates the same results. The analysis could be extended to allow for nonlinearity and heterogeneity as in Siegel (2009). However, this heterogeneity is tangential to our main research question.

[11] The all-pay auction without noise is sometimes seen as an extreme case. However, recent work by Alcalde and Dahm (2010) suggests that there is a whole class of contest success functions with (sufficiently little) noise that have equilibria that are payoff equivalent with those of the all-pay auction without noise.

3 Pure strategy choice of the prize set

We first consider and characterize the subgame perfect equilibria in which all players make a deterministic choice of $j \in B$. In such an equilibrium each player receives an expected payoff of either zero or one, depending on whether the boat chosen by the player attracts more players than it has seats, or not. We first state the results[12]

Proposition 1 *(i) If $kb \geq n$, then all pure strategy equilibria yield a payoff equal to 1 to each player. (ii) If $kb < n$, then, for given b and k, the pure strategy equilibria with the highest total welfare have $n_i = k$ for $b-1$ boats and $n_i = n-(b-1)k$ for one boat. (iii) If a pure strategy equilibrium with the highest total welfare is chosen for a given $m \equiv kb < n$, then the welfare is higher if there are more boats with fewer seats. (iv) A pure strategy equilibrium exists in which all players have a zero payoff if and only if $n \geq b(k+1)$.*

Proof. (i) For $kb \geq n$ quasi-partitions of players among the b boats with $n_j \leq k$ for all $j = 1, ..., b$ exist. Let $(N_1, ..., N_b)$ be such a quasi-partition. In stage 2 each player receives a payoff of 1 given this quasi-partition. Suppose all but player i make their choices in stage 1 according to this quasi-partition deterministically, and let $i \in N_j$ in the candidate equilibrium quasi-partition. If i chooses boat j, i's payoff is equal to 1. If i chooses a different boat j', then i's payoff is at most equal to 1. Moreover, we need to rule out inefficient pure strategy equilibria. Suppose there is a pure strategy equilibrium with a lower payoff. Then this implies that the quasi-partition in this equilibrium has a set N_j with $n_j > k$ and a set N_s with $n_s < k$. In this case the deterministic choice of a player i in the set N_j yields a lower payoff to i than if the player chooses $s \in B$. Hence, the quasi-partition with N_s and N_j is not an equilibrium outcome with deterministic choices of boats.

(ii) For any quasi-partition $(N_1, ..., N_b)$, let $\hat{B}$ be the set of boats j for which $n_j \leq k$ and $\hat{B} \backslash B$ the set of boats for which $n_j > k$. Then the total welfare is equal to $\sum_{j \in \hat{B}} n_j$, as all players at boats with $n_j > k$ receive zero payoff and all other players receive a payoff of 1. The efficiency benchmark is a quasi-partition $(N_1, ..., N_b)$ with

[12] In addition to the pure strategy equilibria and the symmetric mixed strategy equilibrium there is a large set of equilibria in which some players choose pure strategies and other players randomize. There is also a large set of equilibria in which some or all players randomize over strict subsets of the set of lifeboats.

$n_1 = ...n_{b-1} = k$ and $n_b = n-(b-1)k$. This benchmark is also an equilibrium. Consider a player i who joined set N_j for $j \neq b$. This player has a payoff equal to 1 in the candidate equilibrium. If the player chooses any other set, his payoff drops from 1 to zero, as the number of players in this set will then exceed the number k. Similarly, consider a player i who joined set N_b in the candidate equilibrium and has a payoff of 0 there. If this player chooses any other $j \neq b$, the player's payoff is still zero. Hence, staying at the boat that is overbooked is not worse than any of the alteratives.

(iii) For given $kb = m < n$, the maximum welfare in (ii) as a function of k can be written as $(b-1)k = m-k$. This is a strictly decreasing function in k.

(iv) We consider a quasi-partition $(N_1, ..., N_b)$ of players with $n_j \geq k+1$ for all $j \in B$. The condition $n \geq b(k+1)$ is sufficient for such a quasi-partition to exist. We show that this quasi-partition also emerges as a coordination equilibrium. Suppose that all players choose their boat according to the quasi-partition. Consider now one single player i who chooses boat j in the candidate equilibrium. This player receives a payoff of zero if he chooses j. If he chooses any other boat j', the number of players at that boat becomes $n_{j'}+1 > k$. Hence, i's payoff from all other choices is also zero. Since $n \geq b(k+1)$ is also necessary for a quasi-partition of players with $n_j \geq k+1$ for all $j \in B$ this condition is also necessary. ■

Intuitively, if the total number of seats is at least as great as the number of players, there are functions that map the set of players to the set of seats that are one-to-one (injective mappings); no player needs to compete with another player for his seat, and each of these mappings is a subgame perfect equilibrium. Evidently, each of these allocations is efficient. Conversely, any mapping which is not injective, i.e., under which two or more players have to fight over seats, cannot be an equilibrium if $m \equiv kb \geq n$, as players who face competition for their respective seat can simply switch to another boat that has seats that would remain empty.

If, instead, $m \equiv kb < n$, then as long as $b \geq 2$, multiple payoff nonequivalent pure strategy equilibria exist, all having some set of players obtaining an equilibrium payoff of zero. The aggregate payoff of a pure strategy equilibrium depends inversely on the number of full boats to which the excess $n-m$ of players over seats is allocated. The number of players receiving a payoff of one (and hence, the aggregate payoff) is maximized within the set of pure strategy equilibria in an equilibrium that coordinates the strategies of players to place the excess $n-m$ of players over seats into a single boat, keeping all other boats at exact capacity k.[13] The number of players receiving a payoff of one is minimized within the set in an equilibrium that disperses each of the excess $n - m$ players to a different full boat until all boats have at least $k + 1$ players and then arbitrarily after that. In this case excess players are allocated in such a way as to "spoil the pot" for the most boats.

Proposition 1 also implies that, if players can coordinate on the most efficient equilibrium in the lifeboat problem, then lifeboats should be as small as possible.[14] All but one lifeboat should be just filled, and all excess passengers should come together at the one remaining lifeboat (and if the lifeboats were to be of heterogenous size, all excess passengers should come together at the smallest lifeboat). The welfare loss occurring due to the competition for seats is minimized in this case, and equal to the benefit that would emerge from this last lifeboat in the absence of competition for seats. Accordingly, if the total number of seats across all lifeboats is given and the cost of provision of seats is independent of the composition on smaller or larger boats, it is best to build the lifeboats as small as possible. One-person lifeboats (or life vests) are optimal. Of course, coordination on the most efficient equilibrium is a heroic assumption in this context. In particular, if players coordinate on more symmetric equilibria, they are more likely to end up in a situation in which there are more than k players at each boat and their total payoff is zero. This is the implication of part (iv) in the proposition.

It is interesting to consider the robustness of these equilibrium results for different contest success functions. If the rent per player is a strictly decreasing function of the number of excess passengers - an assumption that is fulfilled for many types of contest success functions - and if the number of passengers is $n = hb$ with h being an

[13] Note also that if players were given an outside option which offered a certain payoff equal to that received when not obtaining a seat ("going down with the ship") there exist asymmetric pure strategy equilibria which are efficient. In these equilibria any set of $n - m$ players choose the outside option with certainty and the m remaining players play pure strategies which generate an equal allocation of players across the boats.

[14] Here and in what follows we use the terms "coordinate" and "coordinated equilibrium" for the type of subgame perfect equilibrium that is in pure strategies in stage 1 and characterized in Proposition 1 and to distinguish them from the equilibria with mixing in stage 1 that will be introduced later. Of course, for a choice of an equilibrium with mixing also some coordination in the selection of equilibrium is necessary.

integer, the symmetric equilibrium in which all boats are equally crowded, as in the equilibrium characterized in (iv), is induced as a unique coordination equilibrium. We show this in the appendix.

Of course, the coordination that underlies the pure strategy equilibria in Proposition 1 is a strong assumption. We therefore view the symmetric but uncoordinated equilibria as more plausible outcomes and consider them next.

4 Mixed strategy choice of the prize set

We now consider subgame perfect equilibria in which players are unable to coordinate on a pure strategy equilibrium and where they randomize independently and symmetrically in stage 1. The following proposition characterizes the equilibrium payoffs as a function of b, k and n in this case.

Proposition 2 *The payoff of each player in the symmetric equilibrium without coordination is*

$$\pi^*(n,b,k) = \sum_{h=0}^{k-1} \binom{n-1}{h} (1-\frac{1}{b})^{(n-1)-h} \left(\frac{1}{b}\right)^h . \tag{1}$$

Proof. Consider the payoffs of players who arrive at a given boat. Let n_j be the number of players arriving at boat j which has k seats. Recall from section 2: if $n_j \leq k$, the players need not fight for a seat, hence, the payoff for each of these players is equal to the value of having a seat, which we normalized to $v = 1$. If $n_j > k$, then the equilibrium payoff of players in an all-pay auction is zero if the number of (symmetric) players exceeds the number of (identical) prizes (Lemma 1). Hence, players arriving at boats with sufficient supply have a payoff of 1 and players arriving at a boat with excess demand have a payoff of zero.

Turn now to stage 1. Suppose that there is an equilibrium in which all players $2, 3, ..., n$ randomize symmetrically and independently, i.e., each player goes to each of the boats with the same probability. Then this probability is $\frac{1}{b}$ for each player for each boat. The probability that h of the other $(n-1)$ players show up at boat j as well is

$$\binom{n-1}{h} (1-\frac{1}{b})^{(n-1)-h} \left(\frac{1}{b}\right)^h \tag{2}$$

for $h \leq n-1$, and this probability is the same for all boats. Accordingly, the probability that no conflict takes place at boat j if player 1 goes to this boat is

$$\pi(n,b,k) = \sum_{h=0}^{k-1} \binom{n-1}{h} (1-\frac{1}{b})^{(n-1)-h} \left(\frac{1}{b}\right)^h . \tag{3}$$

Symmetric randomization is optimal for player 1 given that all other players randomize symmetrically. This follows from the fact that the expected payoff from going to any of the b boats is exactly the same. Moreover, if all n players independently randomize across all boats b, then their expected payoffs are equal to this value. ■

Equation (3) shows how the payoff in the symmetric mixed strategy equilibrium depends on b and k for a given total capacity $m = kb$. We first consider the extremes. Let $b = 1$ and $k = m$. All players will approach the one and only lifeboat. Hence, the aggregate payoff from this is large and equal to n if and only if $n \leq k = m$, but the aggregate payoff will drop to zero for all $n > k = m$. Compare this with the other extreme case with lifeboats with the smallest possible capacity of $k = 1$, but a number of lifeboats equal to $b = m$, which generates the same total seating capacity. In this case the expected payoff to each player i from approaching a randomly chosen boat j is equal to the probability that no other player approaches this boat, which is equal to $(1 - (1/b))^{n-1}$, and this is each player's equilibrium payoff in this case. This payoff is strictly smaller than 1 even if there are many more boats than there are players, but it is also strictly positive when $m < n$, i.e., if there is an absolute shortage of seats.

This discussion motivates the following characterization of the optimal composition (b, k) for a given total number of seats m and players n in the uncoordinated symmetric equilibrium.

Proposition 3 *Consider the symmetric equilibrium in Proposition 2. Let $m, n > 2$. (i) If $m \geq n$, then $k = m$ and $b = 1$ maximizes the expected payoff of all players. If $m < n$, then $k = m$ and $b = 1$ is dominated by all other combinations (b, k) with $kb = m$. (ii) If $m < n$, and m is even, then the optimum is an interior solution ($k \geq 2$ and $k < m$) if*

$$\left(\frac{m-1}{m-2}\right)^{n-2} < \frac{m+2n-4}{m-1} \tag{4}$$

For any given $m > 1$ there is a sufficiently large n such that (4) is violated.

Proof. The first part of the proposition has already been established by the discussion of the extreme cases $m = b$ and $m = k$. The second part of the proposition can be established by comparing payoffs for $(k = 1;\ b = m)$ and $(k = 2;\ b = (m/2))$. We already derived the payoff for $k = 1$ as $(1-(1/b))^{n-1} = (1-(1/m))^{n-1}$. To derive the payoff for $k = 2$ and $b = m/2$, note that the probability that no other player arrives at boat j is $(1-(1/b))^{n-1} = (1-(2/m))^{n-1}$. Moreover, the probability that just one other player arrives at boat j is equal to $(n-1)(1-(2/m))^{n-2}(2/m)$. Hence, a player's equilibrium payoff for $(k = 1;\ b = m)$ is lower than for $(k = 2;\ b = (m/2))$ if

$$(1-(1/m))^{n-1} < (1-(2/m))^{n-1} + (n-1)(1-(2/m))^{n-2}(2/m). \tag{5}$$

The condition (5) can be transformed into (4). ■

The first result in Proposition 3 suggests that, if there is no absolute shortage of boat seats, efficiency can be achieved by having only one boat – as this essentially solves the coordination problem. If there is an absolute shortage of seats the lack of coordination has two effects. On the one hand it may cause a welfare loss due to inefficient use of capacity: with some probability some boats have strictly fewer passengers than seats. On the other hand, passengers in boats which do not have more passengers than seats enjoy a positive rent. If there are more but smaller boats, this changes the relative importance of these two effects.

The comparative static properties of this equilibrium are numerically straightforward. For instance, the additional payoff derived from an additional boat of given boat-size k is $\pi^*(n, b+1, k) - \pi^*(n, b, k)$, and using (1) it can be written explicitly as

$$\sum_{h=0}^{k-1} \binom{n-1}{h} \left[(1-\frac{1}{b+1})^{(n-1)-h} \left(\frac{1}{b+1}\right)^h - (1-\frac{1}{b})^{(n-1)-h} \left(\frac{1}{b}\right)^h \right]. \tag{6}$$

Similarly, the additional payoff derived from an additional seat in each of the boats is $\pi^*(n, b, k+1) - \pi^*(n, b, k)$, and using (1) it can be written explicitly as

$$\binom{n-1}{k} (1-\frac{1}{b})^{(n-1)-k} \left(\frac{1}{b}\right)^k. \tag{7}$$

We can also look at the trade-off between many small boats versus few large boats for a given overall capacity bk. The two extreme cases are $(b, k) = (m, 1)$ and $(b, k) = (2, (m/2))$. Let the number of passengers be r times the total number of seats: $n = rm$, with $r \geq 2$. Consider first $(b, k) = (m, 1)$. The probability that a player is the only player arriving at a given boat in the symmetric equilibrium is $(1-(1/m))^{rm-1}$. As m increases

$$\lim_{m\to\infty}(1-(1/m))^{rm-1}=e^{-r}>0. \tag{8}$$

Hence, even if the number of passengers vastly exceeds the number of seats, as long as the ratio of passengers to seats remains fixed, the probability that a single passenger arrives at an empty seat is quite substantial. In contrast, if there are only two boats with joint capacity of m, and the number of passengers is r times as large as the total number of seats, as m increases without bound the probability that one of the boats is approached by less than $m/2$ passengers is driven down to zero.[15]

Table 1 provides further insight into the comparative static properties of the equilibrium payoff. It shows the equilibrium payoff to a representative player in the symmetric mixed strategy equilibrium for different values of b and k and for $n=100$. Each cell in the table provides the payoff in the case where the number of seats per boat is given by the corresponding row value of k and the total number of seats m is given by the corresponding column value. The number of lifeboats can be easily derived by dividing the corresponding value of m given by the column by the value of k appearing in the row. Blank cells correspond to parameter values for which m/k or m/b is not an integer, so that this number cannot be obtained by utilizing equally sized lifeboats with k seats each. As can be easily seen, for values of m greater than $n=100$, payoffs are monotonically increasing in the number of seats per boat until a point is reached where the number of seats per boat equals the number of players. The table also illustrates the large welfare losses that may be realized when players choose their boat due to the "coordination failure" arising from the randomization generated by the symmetric mixed strategy equilibrium. Even if there are double the number of seats as players ($m=200$ and $n=100$), if these seats are configured in 200 one-seat boats, the welfare is only approximately 60% of that arising in with a single boat of 200 seats (or two boats with 100 seats). If there are exactly 100 seats the welfare with 100 single-seat boats is approximately 37% of that with a single 100-seat

[15] The probability that a given boat is approached by no more than $m/2$ passengers is equal to the cumulative distribution function of the binomial distribution with parameters rm and $1/2$, evaluated at $m/2$. Using normal approximation to the binomial cumulative distribution with a continuity correction this probability may be written as $\Phi((1-r)(\frac{m}{r})^{1/2}+(rm)^{-(1/2)})$ which (since $1-r<0$) approaches 0 as $m\to\infty$.

boat. Moreover, even with five twenty-seat boats, this welfare rises only to about 48% of the optimal welfare.

k \ m	60	80	90	96	100	120	200
1	**.189**	.288	.331	.355	.370	.437	.609
2	.154	**.289**	.351	.386	.409	.507	.739
3	.122	–	**.355**	.399	–	.549	–
4	.097	.265	–	.405	.437	.579	.863
5	.077	.252	.351	–	.445	.604	.897
6	.061	–	.347	.410	–	.625	–
8	–	.215	–	**.411**	–	.660	.955
10	.023	.193	.328	–	.464	.689	.973
12	.014	–	–	.409	–	.715	–
20	.000	.109	–	–	.480	.794	.998
100	–	–	–	–	**1**	–	**1**

Table 1: Payoffs to a representative player for $n = 100$. Blank spaces indicate cases where m is not perfectly divisible by k.

When the total number of seats is strictly lower than the number of players, there is in a sense an optimal amount of coordination failure that may or may not lead to interior configurations of seats per boat. When m is small relative to n, a player can only expect to obtain a positive payoff if the players are sufficiently unequally spread across boats and a player is lucky enough to find himself in a boat that receives a small number of people. In this case the coordination failure resulting from randomization is valuable and the optimal configuration of seats is $b = m$ and $k = 1$. This is indicated in Table 1 in the case where $m = 60$, where the bold entry corresponding to $k = 1$ shows a maximum expected payoff of .189. For $m = 60$, payoffs are decreasing as the number of seats per boat increases. Starting at a value of m slightly below eighty, the welfare maximizing amount of coordination failure decreases sufficiently to generate an interior solution. For $m = 80$, the payoff from setting $k = 2$ and $b = 40$ slightly improves upon $k = 1$ and $b = 80$ and this becomes the optimal configuration, yielding a payoff of .289. Payoffs decrease in k beyond $k = 2$ in this case. For $m = 90$ the optimal configuration is $k = 3$ and $b = 30$, with payoffs increasing in k for lower values of k and decreasing in k for higher values of k. Finally, when the number of players exceeds the number of seats by a number that is very small relative to the number of players ($n = 100, m = 96$) too much coordination failure leads to wasted capacity and the optimal number of seats per boat rises to $k = 8$, corresponding to $b = 12$. In this case, despite the fairly small shortfall in seats, welfare is .411, less than half of the optimum with one seat per passenger ($n = 100, m = 100$) but substantially higher than that which would arise at either extreme when $m = 96$: when $k = 1$and $b = 96$ welfare is .355 and when $b = 1$ and $k = 96$ welfare is zero.

Note that there are also other asymmetric equilibria, some involving deterministic and mixed strategies. For $kb < n$ we illustrate this by characterizing three more intermediate types of equilibrium. First, one interesting type of equilibrium has a partition with $b - m$ boats being just filled to their capacity k, and the remaining $m \geq 2$ boats attracting all remaining $n - (b - m)k$ players with equal probability.[16] For $m = 1$ this degenenerates to the efficient coordination equilibrium in part (ii) of Proposition 1, and for $m = b$, this yields the symmetric equilibrium with mixing as in Proposition 2. As can be shown, for this set of equilibria, the average payoff of players decreases in m. However, as has been argued in the context of coordination equilibria, the existence of equilibria for $m < b$ is sensitive to the choice of the contest success function. For a contest success function that induces a payoff that is strictly decreasing in the number of excess passengers that arrive at a given boat, these asymmetric outcomes are not robust. Second, there can also be equilibria in which the players who randomize do not randomize symmetrically across all boats. For example, let there be n players and b boats with capacity k, and let n and b be even. Then there is an equilibrium in which $n/2$ of the players randomize symmetrically across the boats 1 to $b/2$, and the other half of the players randomize across

[16] To see that this is an equilibrium, note that all players employing a deterministic strategy have the highest possible payoff of 1, whereas players who randomize between the m remaining boats have a payoff equal to $\pi^*(n - (b - m)k, m, k) > 0$ at any of the m remaining boats, and a payoff of zero if they deviate and choose one of the $b - m$ boats that are exactly filled.

the second half of the boats. Their equilibrium payoffs are $\pi^*(n/2, b/2, k)$.[17] Third, for $k \geq 2$ there is a set of equilibria in which bc players each choose deterministically one boat in a way such that every boat has $c < k$ of these players, and all other players randomize symmetrically among the b boats. The probability that no more than k players show up at a boat i in this case is equal to the probability that $k - c$ or less of the $(n - bc)$ randomizing players show up at this boat. For this group of $(n - bc)$ randomizing players, the problem is equivalent to the lifeboat problem with this number of players and b boats with capacity $k - c$. Accordingly, the equilibrium payoff is $\pi^*((n - bc), b, (k - c))$, and this is also the payoff for each of the players who deterministically choose this boat.

As the welfare in the equilibrium with pure strategy boat-selection choices in stage 1 generates full efficiency for $n \leq kb$, but not for the symmetric equilibrium with randomization, the coordinated equilibrium is superior to the fully uncoordinated symmetric equilibrium for $n \leq kb$. The comparison is much less clear for $n > kb$, as there is a wide variety of equilibria in both deterministic and stochastic boat-selection choices as well as hybrid equilibria. In particular, coordinated equilibria need not yield higher total welfare than uncoordinated equilibria, as any of the uncoordinated equilibria we have considered has higher expected payoffs for the players than the symmetric fully coordinated equilibrium of type (iv). Moreover, even a type (ii) coordinated equilibrium that yields highest total welfare yields an expected payoff of zero to a non-empty subset of players. Since these players earn a positive expected payoff in uncoordinated equilibria, no coordination equilibrium Pareto dominates any of the uncoordinated equilibria if there are more passengers than seats.

5 Conclusions

Players often have to choose which contest to enter, not knowing the decisions of other players. As a result, some players may find themselves in a situation with many players competing for few prizes, in which competition is strong, or alternatively they may end up in an environment with few other competitors compared to the number of prizes. In this article we highlight the role of the partitioning of prizes into different

[17] A proof follows the lines of proof in Proposition 2 for each of the two subgroups and from the observation that none of the players can gain from switching to the other subset of boats.

sets and the importance of the process by which players choose the set of prizes over which they compete. We characterize the set of pure strategy equilibria and the symmetric equilibrium that is typically in mixed strategies. We find that when total capacity is too small to accommodate all users, a partition of that capacity into smaller subsets ("boats") can increase welfare if individuals must simultaneously select a set of prizes ("seats") over which they compete. This result holds for both types of equilibria that we consider: both coordinated equilibria in pure strategies and the symmetric mixed strategy equilibrium have higher welfare with the partition of prizes. This result uncovers a general principle which may apply universally to a large number of problems that range from education choices among different, mutually exclusive alternatives to choices of territory among animals. Here we focus on a particular type of interaction in case capacity falls short of demand: an all-pay contest without noise. Other allocation mechanisms in which players discontinuously suffer if the number of players arriving at a "boat" exceeds the number of "seats" are likely to generate similar results.

6 Appendix

In this appendix we consider a situation with

$$b(k+r) = n \text{ for some } r \in N \tag{9}$$

and characterize the set of equilibria with any contest success function in the subgame in stage 2 for which the expected equilibrium payoff per player in the contest with h players and k identical prizes is strictly decreasing in h for $h \geq k$ and constant in h for $h \leq k$, where h and k are natural numbers. We show:

Proposition 4 *For any equilibrium in pure strategies the number of passengers at each boat b_j is equal to $n_j = k + r$.*

Proof. Consider any partition of players $n_1, ..., n_b$ yielding an expected payoff to each player who has chosen boat i of $u(n_i)$. Condition (9) implies that either $n_1 = n_2 = ... = n_b$, or there are at least two boats i and j with $n_i > k + 1$ and $n_i \geq n_j + 2$. In turn, this implies that the partition cannot be an equilibrium, as each player in boat i could increase his expected payoff from $u(n_i)$ to $u(n_j + 1)$ by defecting to boat j. ■

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